Cuaderno de notas de trabajo

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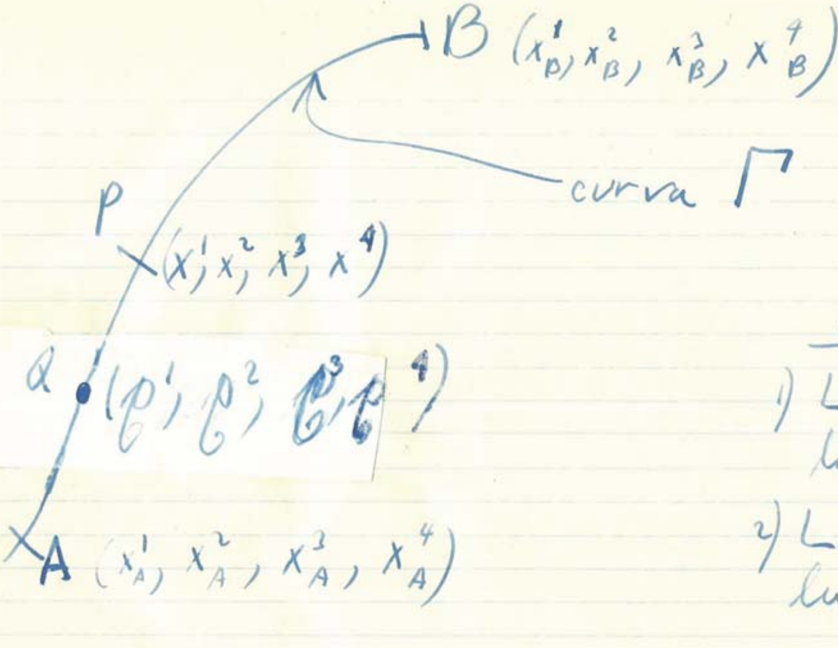
Cuaderno Sn 5



Notación:

1) Letras góticas para las funciones de Q.
2) Letras romanas para las funciones de P.

$G_{ij}(x^1, x^2, x^3, x^4)$; $\mathfrak{g}_{ij}(\mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^3, \mathfrak{p}^4)$

G_{ij}, $\mathfrak{g}_{ij}$ tensor de dos índices (covariantes)

$$\boxed{J = \int_A^B \left[\int_A^P \mathfrak{g}_{ij}\, d\mathfrak{p}^j\right] dx^i}$$

$$\delta J = \int_A^B \left\langle \left\{\delta \left[\int_A^P \mathfrak{g}_{ij}\, d\mathfrak{p}^j\right]\right\} dx^i + \left[\int_A^P \mathfrak{g}_{ij}\, d\mathfrak{p}^j\right] \delta(dx^i) \right\rangle$$

$$(\delta J)_1 = \int_A^B \left\langle \left\{\delta \left[\int_A^P \mathfrak{g}_{ij}\, d\mathfrak{p}^j\right]\right\} dx^i \right\rangle$$

$$(\delta J)_2 = \int_A^B \left\langle \left[\int_A^P \mathfrak{g}_{ij}\, d\mathfrak{p}^j\right] \delta(dx^i) \right\rangle$$

$$\cancel{\delta J = \int_A^B \left[\int_A^P \frac{\partial g_{ij}}{\partial p^k}\,\delta p^k\, dp^j\right] dx^i}$$

$$\cancel{\delta J = \iint}$$

$$\delta\int_A^P g_{ij}\, dp^j = \int_A \left[\frac{\partial g_{ij}}{\partial p^k}\,\delta p^k\, dp^j + g_{ij}\, d(\delta p^j)\right]$$

$$\delta\int_A^P g_{ij}\, dp^j = g_{ij}\,\delta p^j \Big|_A^P + \int_A^P \left[\frac{\partial g_{ij}}{\partial p^k}\,\delta p^k\, dp^j - \frac{\partial g_{ij}}{\partial p^p}\,\delta p^i\, dp^p\right]$$

$$\boxed{\delta\int_A^P g_{ij}\, dp^j = G_{ij}\,\delta x^j + \int_A \left[\frac{\partial g_{ia}}{\partial p^b} - \frac{\partial g_{ib}}{\partial p^a}\right]\delta p^b\, dp^a}$$

$$\cancel{\delta J_1 = \int_A^B G_{ij}\,\delta x^j\, dx^i + \int_A^P \left\langle \frac{\partial g_{ia}}{\partial p^b} - \frac{\partial g_{ib}}{\partial p^a}\right.}$$

$$\boxed{\delta J_1 = \int_A^B G_{ij}\,\delta x^j\, dx^i + \int_A^B \left\langle \left\{\int_A \left[\frac{\partial g_{ia}}{\partial p^b} - \frac{\partial g_{ib}}{\partial p^a}\right]\delta p^b\, dp^a\right\} dx^i \right\rangle}$$

$$(\delta J)_2 = \int_A^B \left\langle \left[\int_A^P g_{ij}\, d\rho^j_\sigma\right] d(\delta x^i) \right\rangle$$

$$(\delta J)_2 = \left[\int_A^P g_{ij}\, d\rho^j_\sigma\right] \delta x^i \Big|_A^B - \int_A^B G_{ij}\, dx^j\, \delta x^i$$

$$\boxed{\delta J_2 = - \int_A^B G_{ij}\, dx^j\, \delta x^i}$$

Hipótesis: G_{ij}, g_{ij} tensor simétrico.

$$\boxed{\delta J = \int_A^B \left\langle \int_A^P \left[\frac{\partial g_{ij}}{\partial \rho^k_\sigma} - \frac{\partial g_{ik}}{\partial \rho^j_\sigma}\right] \delta \rho^k_\sigma\, d\rho^j_\sigma \right\rangle dx^i}$$

$$\boxed{J = \int_A^B \left\langle \int_A^P g_{ij}\, d\rho^j_\sigma \right\rangle dx^i}$$

Δ_{ij}, h_{ij}, v^i, dx^i, $\boxed{\cancel{\frac{\partial h_{ij}}{\partial x^k}}}$

$\Delta^{ij}h_{ij}$

$$a_i = \frac{\partial h_{ij}}{\partial x^k} \cancel{v^j} v^j v^k - \frac{\partial h_{jk}}{\partial x^i} v^j v^k$$

$$h^{ip} a_i = \underline{h^{ip} \frac{\partial h_{ij}}{\partial x^k}} v^j v^k - h^{ip} \frac{\partial h_{jk}}{\partial x^i} v^j v^k$$

$$\frac{\partial h_{ij}}{\partial x^k} v^j - \frac{\partial h_{jk}}{\partial x^i} v^j = T_{ik}$$

$$\cancel{\partial h} \; h^{ik}\left(\frac{\partial h_{ij}}{\partial x^k} \cancel{v^j} - \frac{\partial h_{jk}}{\partial x^i}\right) v^j = h^{ik} T_{ik}$$

$$S = \int_A^B \sqrt{\Delta_{ij}\, dx^i dx^j} \; ; \qquad \Sigma = \int_A^B \sqrt{h_{ij}\, dx^i dx^j}$$

$$J = S - \Sigma . \qquad \delta J = 0 .$$

$$\delta J = \int_A^B \frac{\Delta_{ij}\, dx^i\, d(\delta x^j)}{ds} - \int_A^B \frac{h_{ij}\, dx^i\, d(\delta x^j)}{\sqrt{h_{ij}\, dx^i dx^j}}$$

$$\delta J = -\int_A^B a_j\, \delta x^j\, ds + \int_A^B \frac{d}{ds}\left(\frac{h_{ij}\, v^i}{\sqrt{h_{i\beta}\, v^i v^\beta}}\right)(\delta x^j)$$

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$$\Delta^{kl} \frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} v^i v^j$$

$$\frac{\partial h_{ij}}{\partial x^r} v^i v^j$$

$$\delta J = -\int_A^B a_j \delta x^j ds + \int_A^B \frac{\sqrt{h_{ij} v^i v^j} \quad \partial h_{ij}}{\sqrt{h_{ij} v^i v^j}}$$

$$S_{(n)} = h^{i}_{p_1} h^{p_1}_{p_2} h^{p_2}_{p_3} \cdots h^{p_{n-1}}_{p_n} h^{p_n}_{j} v^{a}_{i} v^{j}_{a}$$

$$v^i = \frac{dx^i}{ds}$$

$$\cancel{ds = \sqrt{\Delta_{ij} v}} \qquad ds = \sqrt{\Delta_{ab}\, dx^a dx^b}$$

$$\delta(ds) = \frac{\Delta_{ab}\, dx^a\, d(\delta x^b)}{ds} = \Delta_{ab} \frac{dx^a}{ds} d(\delta x^b)$$

$$\boxed{\delta(ds) = v_a\, d(\delta x^a)}$$

$$\delta v^i = \frac{d}{ds}(\delta x^i) - \frac{dx^i}{ds^2} v_a\, d(\delta x^a)$$

$$\boxed{\delta v^i = \frac{d}{ds}(\delta x^i) - v^i v_a\, d(\delta x^a)}$$

$$\boxed{\delta(v_j) = \frac{d}{ds}(\delta x_j) - v_j\, v_a\, d(\delta x^a)}$$

$$\boxed{\delta\left(h^{p_k}_{p_{k+1}}\right) = \frac{\partial h^{p_k}_{p_{k+1}}}{\partial x^c}\, \delta x^c}$$

$$\delta S_{(n)} = \Big[\frac{\partial h^i_{p_1}}{\partial x^c} h^{p_1}_{p_2} h^{p_2}_{p_3} \cdots h^{p_{n-1}}_{p_n} h^{p_n}_j\, v_i v^j\, \delta x^c$$

$$+\ h^i_{p_1} h^{p_1}_{p_2} h^{p_2}_{p_3} \cdots h^{p_{n-1}}_{p_n} \frac{\partial h^{p_n}_j}{\partial x^c}\, v_i v^j\, \delta x^c$$

$$+ \sum_{k=1}^{n-1} h^i_{p_1} h^{p_1}_{p_2} \cdots h^{p_{k-1}}_{p_k}\, h^{p_{k+1}}_{k+2} \cdots h^{p_n}_j\, v_i v^j \frac{\partial h^{p_k}_{p_{k+1}}}{\partial x^c}\, \delta x^c$$

$$+$$

$$\cancel{b_{ij} v^j} = \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^i} \right) v^j$$

$$\delta \int_A^B h_{ij}\, dx^j = \int \left[\frac{\partial h_{ij}}{\partial x^k} \delta x^k dx^j + h_{ij}\, d(\delta x^j) \right]$$

$$\delta \int_A^B h_{ij}\, dx^j = \int_A^B \left[\frac{\partial h_{ij}}{\partial x^k} dx^j \delta x^k - \frac{\partial h_{ij}}{\partial x^k} dx^k \delta x^j \right]$$

$$\boxed{\delta \int_A^B h_{ij}\, dx^j = \int_A^B \left[\frac{\partial h_{ij}}{\partial x^k} dx^j - \frac{\partial h_{ik}}{\partial x^j} dx^j \right] \delta x^k}$$

$$\left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j}\right) v^j = T_{ik}$$

$$\frac{dT_{ik}}{ds} = \left(\frac{\partial^2 h_{ij}}{\partial x^k \partial x^p} - \frac{\partial^2 h_{ik}}{\partial x^j \partial x^p}\right) v^j v^p + \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j}\right) a^j$$

$$\frac{dT_{ik}}{ds} = \left(\frac{\partial^2 h_{ij}}{\partial x^k \partial x^p} - \frac{\partial^2 h_{ik}}{\partial x^j \partial x^p}\right) v^j v^p + \Delta^{jr}\left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j}\right)\left(\frac{\partial h_{rs}}{\partial x^t} - \frac{\partial h_{st}}{\partial x^r}\right) v^s v^t$$

$$\frac{dT_{ik}}{ds} = \left[\left(\frac{\partial^2 h_{is}}{\partial x^k \partial x^t} - \frac{\partial^2 h_{ik}}{\partial x^s \partial x^t}\right) + \Delta^{jr}\left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j}\right)\left(\frac{\partial h_{rs}}{\partial x^t} - \frac{\partial h_{st}}{\partial x^r}\right)\right] v^s v^t$$

$\Delta^{ij} h_{ij}\,,\quad h_{ij} v^i v^j\,,$

$$(1-\nu z)^{-\frac{1}{2}} = 1 + \frac{1}{2}\nu z + \frac{1}{2}\cdot\frac{3}{4}\nu^2 z^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\nu^3 z^3 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\nu^4 z^4 + \cdots$$

$$(1-\nu^{-1} z)^{-\frac{1}{2}} = 1 + \frac{1}{2}\nu^{-1} z + \frac{1}{2}\cdot\frac{3}{4}\nu^{-2} z^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\nu^{-3} z^3 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\nu^{-4} z^4 + \cdots$$

$$(1-\nu z)(1-\nu^{-1} z)^{-\frac{1}{2}} = \Big[\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}\frac{(2n-1)}{2n}\left(\nu^{n}+\nu^{-n}\right) + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdots$$

$$+ \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\cdot\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-7)}{2n-6}\right)\left(\nu^{n-6}+\nu^{-n}\cdots\right)\Big]$$

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-2)(2n-1)(2n)\,\nu^{-n}}{2^n\cdot n!\cdot 2^n\cdot n!}\left[\begin{array}{l} 1 + \frac{1}{2}\cdot\frac{2n}{2n-1}\nu^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{2n}{(2n-1)}\cdots \\ 1\cdot\nu^{2n} + \frac{1}{2}\cdot\frac{2n}{(2n-1)}\nu^{2n-2} + \frac{1}{2}\cdot\frac{3}{4}\cdots \end{array}\right.$$

$$\cdots + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}\frac{(2n-1)}{2n}\nu^{n}z^{n} + \cdots$$

$$+ \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{2n-2}\frac{(2n-1)}{2n}\nu^{-n}z^{n} + \cdots$$

$$\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}\left(\nu^{n-2}+\nu^{-n+2}\right) + \cancel{4}\cdot\left(\frac{1}{2}\cdot\frac{3}{4}\right)\cdot\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-5)}{(2n-4)}\right)\left(\nu^{n-4}+\nu^{-n+4}\right.$$

$$\left.+6\right) + \cdots$$

$$\cdot\frac{(2n-2)}{(2n-3)}\nu^{4} + \cancel{4}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\,\frac{2n(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}\nu^{6} + \cdots$$

$$\frac{2n}{(2n-1)}\cdot\frac{(2n-2)}{(2n-3)}\nu^{2n-4} + \cancel{4}\,\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{2n(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}\nu^{2n-6} + \cdots$$

$$F\left(\tfrac{1}{2}, -n, \tfrac{1}{2}-n, \gamma^2\right)$$

$$\boxed{F} = \cancel{\frac{\frac{1}{2}\cdot(-n)}{1}}$$

$$F = 1 + \frac{\frac{1}{2}\cdot(-n)}{1\cdot\left(\frac{1}{2}-n\right)}\gamma^2 + \frac{\frac{1}{2}\left(\frac{3}{2}\right)(-n)(-n+1)}{1\cdot 2\cdot\left(\frac{1}{2}-n\right)\left(\frac{3}{2}-n\right)}\gamma^4 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot(-n)(-n+1)(-n+2)}{1\cdot 2\cdot 3\cdot\left(\frac{1}{2}-n\right)\left(\frac{3}{2}-n\right)\left(\frac{5}{2}-n\right)}$$

$$F = 1 + 1\cdot\frac{1}{2}\cdot\frac{2n}{(2n-1)}\gamma^2 + \frac{1\cdot 3}{2\cdot 4}\,\frac{2n(2n-2)}{(2n-1)(2n-3)}\gamma^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{2n(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}$$

$$\cancel{\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 4\cdot 6\cdot 8\cdots 2n}\,\frac{2n(2n-2)(2n-4)\cdots 2}{(2n-1)(2n-3)\cdots 1}\;\gamma^{2n}}$$

$2n-8+2$ $\quad 2n-8+1$ $\quad 2n-(2n-2)+2$

$4 \quad 1$

$$\nu^6 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot(-n)(-n+1)(-n+2)(-n+3)}{1\cdot 2\cdot 3\cdot 4\,\left(\frac{1}{2}-n\right)\left(\frac{3}{2}-n\right)\left(\frac{5}{2}-n\right)\left(\frac{7}{2}-n\right)}\nu^8 + \dots$$

$$\nu^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\,\frac{2n(2n-2)(2n-4)(2n-6)}{(2n-1)(2n-3)(2n-5)(2n-7)}\nu^8 + \dots$$

$$+\frac{\cancel{1\cdot 3\cdot 5\cdots(2n-3)}\cdot 2n\cancel{(2n-2)(2n-6)\cdots 4}}{\cancel{2\cdot 4\cdot 6\cdots(2n-2)}\cdot(2n-1)\cancel{(2n-3)\cdots 3}}\nu^{2n-2}$$

$$P_n(\mu) = \frac{(2n)!\;\nu^{-n}}{2^{n}(n!)^2}\left[F\left(\tfrac{1}{2}, -n, \tfrac{1}{2}-n, \nu^2\right)\right]$$

$$\mu P_n'(\mu) - P_{n-1}'(\mu) = n P_n(\mu) \qquad (35)$$

$$P_{n+1}'(\mu) - \mu P_n'(\mu) = (n+1)P_n(\mu) \qquad (36)$$

$$nP_{n+1}'(\mu) - (2n+1)\mu P_n'(\mu) + (n+1)P_{n-1}'(\mu) = 0 \qquad \times$$

$$\boxed{\mu P_n'(\mu) = \varphi_n}$$

$$\varphi_n - P_{n-1}' - nP_n = 0$$

$$-\varphi_n \qquad - P_{n+1}' - (n+1)P_n = 0$$

$$\cancel{nP_{n+1}' - (n+1)\varphi_n} + (n+1)P_{n-1}' - (n+1)$$

$$\varphi_n - P'_{n-1} \quad \cdot \quad - nP_n = 0$$

$$-\varphi_n \quad \cdot \quad + P'_{n+1} - (n+1)P_n = 0$$

$$-(2n+1)\varphi_n + (n+1)P'_{n-1} + nP'_{n+1} = 0$$

$$\begin{vmatrix} 1 & -1 & -nP_n \\ -1 & 0 & \left(+P'_{n+1} - (n+1)P_n\right) \\ -(2n+1) & (n+1) & +nP'_{n+1} \end{vmatrix} = 0$$

$$n(n+1)P_n + \left(-P'_{n+1} + (n+1)P_n\right)\underbrace{\left[(n+1) - (2n+1)\right]}_{-n} + nP'_{n+1}(-1) = 0$$

$$n(n+1)P_n - n\left(-P'_{n+1} + (n+1)P_n\right) - n\left(P'_{n+1}\right) = 0$$

$$\cancel{-P_n P'_{n+1} = \left[-n(n+1) + n(n+1)P_n\right.}$$

$$(37) \quad (1-\mu^2)P'_n(\mu) = n P_{n-1}(\mu) - n\mu P_n(\mu) \qquad \Big| \; (n+1)$$

$$(37') \quad (1-\mu^2)P'_n(\mu) = (n+1)\mu P_n(\mu) - (n+1)P_{n+1}(\mu) \qquad \Big| \; n \quad \Big\} +$$

$$(2n+1)(1-\mu^2)P'_n(\mu) = n(n+1)P_{n-1}(\mu) - n(n+1)P_{n+1}(\mu)$$

$$(2n+1)(1-\mu^2)P'_n(\mu) = n(n+1)\left[P_{n-1}(\mu) - P_{n+1}(\mu)\right]$$

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$$\cancel{\mu^2-1} \int_{-1}^{+1} (2n+1)(1-\mu^2) P_{n+1} P_n' \, d\mu = n(n+1)\int_{-1}^{+1} P_{n+1} P_{n-1} \, d\mu - n(n+1)\int_{-1}^{+1} P_{n+1}^2 \, d\mu$$

$$\int_{-1}^{+1} (1-\mu^2) P_{n+1} P_n' \, d\mu = -\frac{n(n+1)}{(2n+1)} \int_{-1}^{+1} P_{n+1}^2 \, d\mu$$

$$\int_{-1}^{+1} (\mu^2-1) P_{n+1}(\mu) P_n'(\mu) \, d\mu = \frac{n(n+1)}{(2n+1)} \frac{1}{(2n+3)} = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

n even

$$\int_0^1 \mu^m P_n \, d\mu = \frac{m(m-2)(m-4)\cdots(m-n+2)}{(m+n+1)(m+n-1)(m+n-3)\cdots(m+1)}$$

$$\int_0^1 \mu^{m-1} P_{n-1} \, d\mu = \frac{(m-2)(m-4)\cdots(m-n+2)}{(m+n-1)(m+n-3)\cdots(m+1)}$$

$$(m+n+1)\int_0^1 \mu^m P_n(\mu) \, d\mu = m \int_0^1 \mu^{m-1} P_{n-1}(\mu) \, d\mu$$

$$\cancel{\frac{m}{m+n+2}}$$

n odd

$$\int_0^1 \mu^m P_n \, d\mu = \frac{(m-1)(m-3)\cdots(m-n+2)}{(m+n+1)(m+n-1)(m+n-3)\cdots(m+2)}$$

$$\int_0^1 \mu^{m-1} P_{n-1} \, d\mu = \frac{(m-1)(m-3)(m-5)\cdots(m-n+2)}{(m+n-1)(m+n-3)\cdots(m+2)\, m}$$

$$(m+n+1)\int_0^1 \mu^m P_n(\mu)\, d\mu = m \int_0^1 \mu^{m-1} P_{n-1}(\mu)\, d\mu$$

n even $\quad (n-2, \text{ even})$

$$\int_0^1 \mu^m P_n(\mu)\, d\mu = \frac{m(m-2)(m-4)\cdots(m-n+2)}{(m+n+1)(m+n-1)\cdots(m+1)}$$

$$\int_0^1 \mu^m P_{n-2}(\mu)\, d\mu = \frac{m(m-2)\cdots(m-n+4)}{(m+n-1)(m+n-3)\cdots(m+1)}$$

$$(m+n-1)\int_0^1 \mu^m P_n(\mu)\, d\mu = (m-n+2)\int_0^1 \mu^m P_{n-2}(\mu)\, d\mu$$

n odd (n-2, odd)

$$\int_0^1 \mu^m P_n(\mu)\,d\mu = \frac{(m-1)(m-3)\cdots(m-n+2)}{(m+n+1)(m+n-1)\cdots(m+2)}$$

$$\int_0^1 \mu^m P_{n-2}(\mu)\,d\mu = \frac{(m-1)(m-3)\cdots(m-n+4)}{(m+n-1)(m+n-3)\cdots(m+2)}$$

$$(m+n+1)\int_0^1 \mu^m P_n(\mu)\,d\mu = (m-n+2)\int_0^1 \mu^m P_{n-2}(\mu)\,d\mu$$

$$\cancel{\frac{d^2x^i}{ds^2} = \left(\partial h \right.}$$

$$a_i = \left(\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{jk}}{\partial x^i} \right) \frac{dx^j}{ds} \frac{dx^k}{ds}$$

$$a_i = \left(\frac{\partial h_{ij}}{\partial x^k} \frac{dx^k}{ds} \right) \frac{dx^j}{ds} - \frac{\partial h_{jk}}{\partial x^i} \frac{dx^j}{ds} \frac{dx^k}{ds}$$

$$a_i = \left(\frac{d}{ds} \left(h_{ij} \right) \right) \frac{dx^j}{ds} - \frac{\partial h_{jk}}{\partial x^i} \frac{dx^j}{ds} \frac{dx^k}{ds}$$

$$a_i = \frac{d}{ds} \left\{ h_{ij} \frac{dx^j}{ds} \right\} - \frac{\partial}{\partial x^i} \left\{ h_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} \right) - h_{ij} a^j$$

$$\cancel{\delta \int_A^B h_{ij}\, dx^j = \int_A^B \left[\frac{\partial h_{ij}}{\partial x^k}\, \delta x^k\, dx^j + h_{ij}\, d(\delta x^j)\right]}$$

$$\cancel{\delta \int_A^B h_{ij}\, dx^j =}$$

$$I = \int_A^B h_{jk}\, \frac{dx^j}{d\tau}\, dx^k$$

$$\delta I = \int_A^B \left[\frac{\partial h_{jk}}{\partial x^\ell}\, \delta x^\ell\, \frac{dx^j}{d\tau}\, dx^k + h_{jk}\right.$$

$$a_i + h_{ij} a^j = \frac{d}{ds}\{h_{ij} v^j\} - \frac{\partial}{\partial x^i}\{h_{jk} v^j v^k\}$$

$$(\Delta_{ij} + h_{ij}) a^j = \frac{d}{ds}\{h_{ij} v^j\} - \frac{\partial}{\partial x^i}\{h_{jk} v^j v^k\}$$

$$h_{ij} v^i a^j = v^i \frac{d}{ds}\{h_{ij} v^j\} - \frac{d}{ds}\{h_{jk} v^j v^k\}$$

$$= \cancel{\frac{d}{ds}\{h_{ij} v^i v^j\}} - h_{ij} a^i v^j - \cancel{\frac{d}{ds}\{h_{jk} v^j v^k\}}$$

$$I = \int_0^{\pi} \frac{d\varphi}{1+\varsigma\cos\varphi}$$

$$I = \int_0^{\pi} \frac{d\varphi}{(1+\varsigma)\cos^2\frac{\varphi}{2} + (1-\varsigma)\operatorname{sen}^2\frac{\varphi}{2}}$$

$$I = \int_0^{\pi} \frac{\sec^2\frac{\varphi}{2}\, d\varphi}{(1+\varsigma)+(1-\varsigma)\tan^2\frac{\varphi}{2}}$$

$$t = \tan\frac{\varphi}{2}$$

$$I = 2\int_0^{\infty} \frac{dt}{(1+\varsigma)+(1-\varsigma)t^2}$$

$$I = \frac{2}{1+\varsigma}\int_0^{\infty} \frac{dt}{1+\frac{1-\varsigma}{1+\varsigma}t^2}$$

$$I = \frac{2}{(1+\varsigma)}\sqrt{\frac{1+\varsigma}{1-\varsigma}}\int_0^{\infty} \frac{\sqrt{\frac{1-\varsigma}{1+\varsigma}}\, dt}{1+\left\{\sqrt{\frac{1-\varsigma}{1+\varsigma}}\,t\right\}^2}$$

$$I = \frac{2}{\sqrt{1-\varsigma^2}}\tan^{-1}\left\{\sqrt{\frac{1-\varsigma}{1+\varsigma}}\,t\right\}$$

$$I = \frac{\pi}{\sqrt{1-\varsigma^2}}$$

$$\cancel{\varsigma = \mp\alpha\sqrt{\frac{(z^2-1)}{1-\alpha z}}}$$

$$\varsigma = \mp\alpha\frac{\sqrt{z^2-1}}{1-\alpha z}$$

$$1-\varsigma^2 = 1-\alpha^2\frac{z^2-1}{(1-\alpha z)^2}$$

$$1-\varsigma^2 = \frac{1-2\alpha z+\cancel{\alpha^2 z^2}-\cancel{\alpha^2 z^2}+\alpha^2}{(1-\alpha z)^2}$$

$$\cancel{1-\varsigma^2 = \frac{(1-\alpha z)^2}{(1-\alpha z)^2} = 1}$$

$$\varsigma = \mp \alpha \frac{\sqrt{z^2-1}}{1-\alpha z}$$

$$1-\varsigma^2 = 1 - \alpha^2 \frac{z^2-1}{(1-\alpha z)^2}$$

$$1-\varsigma^2 = \frac{1-2\alpha z+\cancel{\alpha^2 z^2}-\cancel{\alpha^2 z^2}+\alpha^2}{(1-\alpha z)^2}$$

$$\sqrt{1-\varsigma^2} = \frac{\sqrt{1-2\alpha z+\alpha^2}}{1-\alpha z}$$

$$1+\varsigma\cos\varphi = 1 \mp \alpha \frac{\sqrt{z^2-1}}{1-\alpha z}\cos\varphi$$

$$1+\varsigma\cos\varphi = \frac{1-\alpha z \mp \alpha\sqrt{z^2-1}\cos\varphi}{1-\alpha z}$$

$$\mathcal{L} = A_{\alpha_1\alpha_2} V^{\alpha_1} V^{\alpha_2} + A_{\alpha_1\alpha_2\alpha_3\alpha_4} V^{\alpha_1} V^{\alpha_2} V^{\alpha_3} V^{\alpha_4} + \ldots\ldots$$

$$\frac{\partial \mathcal{L}}{\partial V^k} = \Big[A_{k\alpha_2} V^{\alpha_2} + A_{\alpha_1 k} V^{\alpha_1}$$

$$+ A_{k\alpha_2\alpha_3\alpha_4} V^{\alpha_2} V^{\alpha_3} V^{\alpha_4} + A_{\alpha_1 k \alpha_3 \alpha_4} V^{\alpha_1} V^{\alpha_3} V^{\alpha_4}$$

$$+ A_{\alpha_1\alpha_2 k \alpha_4} V^{\alpha_1} V^{\alpha_2} V^{\alpha_4} + A_{\alpha_1\alpha_2\alpha_3 k} V^{\alpha_1} V^{\alpha_2} V^{\alpha_3}$$

$$+ \ldots\ldots$$

$$\frac{d}{ds}\frac{\partial L}{\partial v^k} = \frac{\partial A_{k\alpha_2}}{\partial x^p} v^p v^{\alpha_2} + A_{k\alpha_2} a^{\alpha_2} + \frac{\partial A_{\alpha_1 k}}{\partial x^p} v^p v^{\alpha_1} + A_{\alpha_1 k} a^{\alpha_1}$$

$$+ \frac{\partial A_{k\alpha_2\alpha_3\alpha_4}}{\partial x^p} v^p v^{\alpha_2} v^{\alpha_3} v^{\alpha_4} + A_{k\alpha_2\alpha_3\alpha_4} a^{\alpha_2} v^{\alpha_3} v^{\alpha_4}$$

$$+ A_{k\alpha_2\alpha_3\alpha_4} v^{\alpha_2} a^{\alpha_3} v^{\alpha_4} + A_{k\alpha_2\alpha_3\alpha_4} v^{\alpha_2} v^{\alpha_3} a^{\alpha_4}$$

$$+ \partial A_{\alpha_1\alpha}$$

$$\int_A^B \left\{ \begin{array}{l} \left(\int_A^B F\,dx\right)\left\{\frac{\partial G}{\partial y} - \frac{d}{ds}\left(\frac{\partial G}{\partial y'}\right)\right\} \\ + \left(\int_A^B G\,dx\right)\left\{\frac{\partial F}{\partial y} - \frac{d}{ds}\frac{\partial F}{\partial y'}\right\} \end{array} \right\} \delta y\, dx$$

$$\left\{\int_A^B F\,dx\right\}\left\{\frac{\partial G}{\partial y} - \frac{d}{ds}\frac{\partial G}{\partial y'}\right\} + \left\{\int_A^B G\,dx\right\}\left\{\frac{\partial F}{\partial y} - \frac{d}{ds}\frac{\partial F}{\partial y'}\right\} = 0$$

$$\int_A^B h_{ij}\,dx^j \int_A^C h^i_k\,dx^k = I$$

$$\int_A^B h_{ij}\,dx^j \left\{ \frac{\partial h^i_k}{\partial x^p}\frac{dx^k}{ds} - \frac{d}{ds}\left(h^i_p\right) \right\} + \int_A^B h^i_k\,dx^k \left\{ \frac{\partial h_{ij}}{\partial x^p}\frac{dx^j}{ds} - \frac{d}{ds}\left(h_{ip}\right) \right\} = 0$$

$$\int_A^B$$

$$\delta F\left(\int_A^B\right) = \frac{\partial F}{\partial \int_A^B}\left(\delta \int_A^B\right) = 0 \;,$$

$$\int_0^\pi \frac{d\varphi}{1+\xi\cos\varphi} = \frac{\pi}{\sqrt{1-\xi^2}}$$

$$\xi = \mp \frac{\alpha\sqrt{z^2-1}}{1-\alpha z}$$

$$\cancel{(1-\alpha z)\int_0^\pi \frac{d\varphi}{1-\alpha\sqrt{z^2-1}\cos\varphi}}$$

$$(1-\alpha z)\int_0^\pi \frac{d\varphi}{1-\alpha z \mp \alpha\sqrt{z^2-1}\cos\varphi} = \cancel{(1-\alpha z)}\frac{\pi}{\sqrt{1-2\alpha z + \cancel{\alpha^2 z^2} - \cancel{\alpha^2 z^2} + \alpha^2}}$$

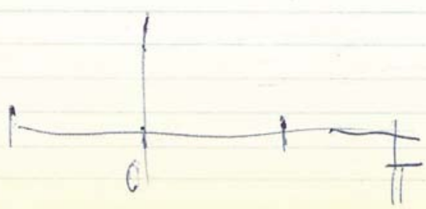
$$\int_0^\pi \frac{d\varphi}{1-\alpha z \mp \alpha\sqrt{z^2-1}\cos\varphi} = \frac{\pi}{\sqrt{1-2\alpha z+\alpha^2}}$$

$$\int_0^{\pi} \frac{d\varphi}{1-\alpha\{z \pm \sqrt{z^2-1}\,\cos\varphi\}} = \frac{\pi}{\sqrt{1-2\alpha z+\alpha^2}}$$

$$\int_0^{\pi}\{z\pm\sqrt{z^2-1}\,\cos\varphi\}^n\, d\varphi = \pi\, P_n(z)$$

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi}\{z\pm\sqrt{z^2-1}\,\cos\varphi\}^n\, d\varphi$$

$$\int_0^{\pi}\cos^{2k+1}\varphi\, d\varphi = \int_0^{\frac{\pi}{2}}\cos^{2k+1}\varphi\, d\varphi + \cancel{\int_{\frac{\pi}{2}}^{0}\cos^{2k+1}\varphi\, d\varphi}$$

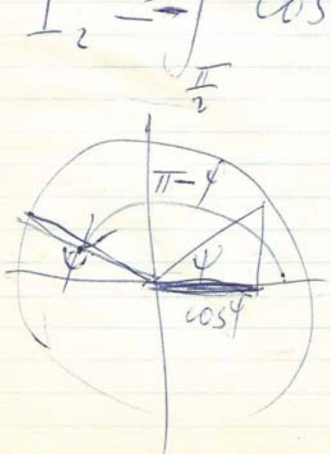


$$\varphi = \pi - \psi$$

$$\int_0^{\pi}\cos^{2k+1}\varphi\, d\varphi = \int_0^{\frac{\pi}{2}}\cos^{2k+1}\varphi\, d\varphi \cancel{+\int_0^{\pi}\cos^{2k+1}\varphi}$$

$$\underbrace{\int_0^{\pi}\cos^{2k+1}\varphi\, d\varphi}_{I} = \underbrace{\int_0^{\frac{\pi}{2}}\cos^{2k+1}\varphi\, d\varphi}_{I_1} + \underbrace{\int_{\frac{\pi}{2}}^{\pi}\cos^{2k+1}\varphi\, d\varphi}_{I_2}$$

$$I_2 = -\int_{\frac{\pi}{2}}^{0}\cos^{2k+1}(\pi-\psi)\, d\psi = +(-)\int_0^{\frac{\pi}{2}}\cos^{2k+1}\psi\, d\psi$$



$$\int_0^{\pi}\cos^{2k+1}\varphi\, d\varphi = 0$$

$$\int e^{\mathcal{L}} ds$$

$$I = \int_A^B e^{\mathcal{L}} ds$$

$$\delta(ds) = \frac{\Delta_{ij}\, dx^i\, d(\delta x^j)}{ds}$$

$$\delta(ds) = \hat{v} \cdot d(\delta \hat{\rho})$$

$$\delta I = \int_A^B \left[e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial x^p} \delta x^p ds + e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial v^q} \delta v^q ds + e^{\mathcal{L}} \left[\hat{v} \cdot d(\delta \hat{\rho}) \right] \right]$$

$$\delta v^q = \cancel{-\frac{dx^q}{ds^2} \hat{v} \cdot} d(\delta \hat{\rho}) + \frac{d}{ds}(\delta x^q) = -v^q \left(\hat{v} \cdot \frac{d}{ds}(\delta \hat{\rho}) \right) + \frac{d}{ds}(\delta x^q)$$

$$\delta I = \int_A^B \left[e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial x^p} \delta x^p + e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial v^q} v^q \left(\hat{v} \cdot \frac{d}{ds}\left(\delta \hat{\rho}\right)\right) + e^{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial m} \frac{d}{ds}\left(\delta x^q\right) + e^{\mathcal{L}} \left\{ \hat{v} \cdot \frac{d}{ds}\left(\delta \hat{\rho}\right)\right\} \right] ds$$